

Idealness of k -wise intersecting families

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Abstract A clutter is *k-wise intersecting* if every k members have a common element, yet no element belongs to all members. We conjecture that, for some integer $k \geq 4$, every k -wise intersecting clutter is non-ideal. As evidence for our conjecture, we prove it for $k = 4$ for the class of binary clutters. Two key ingredients for our proof are Jaeger's 8-flow theorem for graphs, and Seymour's characterization of the binary matroids with the sums of circuits property. As further evidence for our conjecture, we also note that it follows from an unpublished conjecture of Seymour from 1975. We also discuss connections to the chromatic number of a clutter, projective geometries over the two-element field, uniform cycle covers in graphs, and quarter-integral packings of value two in ideal clutters.

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1 Introduction

Let V be a finite set of *elements*, and let \mathcal{C} be a family of subsets of V called *members*. The family \mathcal{C} is a *clutter* over *ground set* V , if no member contains another one [15]. A *cover* of \mathcal{C} is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. Consider for $w \in \mathbb{Z}_+^V$ the dual pair of linear programs

$$\begin{aligned} (P) \quad & \min \quad w^\top x \\ & \text{s.t.} \quad \sum (x_u : u \in C) \geq 1 \quad \forall C \in \mathcal{C} \\ & \quad \quad x \geq \mathbf{0} \\ (D) \quad & \max \quad \mathbf{1}^\top y \\ & \text{s.t.} \quad \sum (y_C : u \in C \in \mathcal{C}) \leq w_u \quad \forall u \in V \\ & \quad \quad y \geq \mathbf{0}. \end{aligned}$$

If the dual (D) has an integral optimal solution for every right-hand-side vector $w \in \mathbb{Z}_+^V$, then \mathcal{C} is said to have the *max-flow min-cut (MFMC) property* [10]. By the theory of *totally dual integral* linear systems, for every MFMC clutter, the primal (P) also admits an integral optimal solution for every cost vector $w \in \mathbb{Z}_+^V$ [16]. This is why the class of MFMC clutters is a natural host to many beautiful *min-max theorems* in Combinatorial Optimization [12]. Let us elaborate.

The *packing number* of \mathcal{C} , denoted $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. Note that $\nu(\mathcal{C})$ is equal to the maximum value of an integral feasible solution to (D) for $w = \mathbf{1}$. Furthermore, the covers correspond precisely to the 0–1 feasible solutions to (P) . The *covering number* of \mathcal{C} , denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. Notice that $\tau(\mathcal{C})$ is equal to the minimum value of an integral feasible solution to (P) for $w = \mathbf{1}$. Also, by Weak LP Duality, $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$. The clutter \mathcal{C} *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$ [28]. Observe that if a clutter is MFMC, then it packs.

Definition 1 ([14]) \mathcal{C} is an *ideal clutter* if the primal (P) has an integral optimal solution for every cost vector $w \in \mathbb{Z}_+^V$.

Ideal clutters form a rich class of clutters, one that contains the class of MFMC clutters, as discussed above. This containment is strict, and in fact, some of the richest examples of ideal clutters are those that are not MFMC [18, 24].

A clutter is *intersecting* if every two members intersect yet no element belongs to every member [4]. That is, a clutter \mathcal{C} is intersecting if $\tau(\mathcal{C}) \geq 2$ and $\nu(\mathcal{C}) = 1$. In particular, an intersecting clutter does not pack, and therefore

is not MFMC. Intersecting clutters, however, may be ideal. For instance, the clutter

$$Q_6 := \{\{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\},$$

whose elements are the edges and whose members are the triangles of K_4 , is an intersecting clutter that is ideal [28]. In fact, Q_6 is the smallest intersecting clutter which is ideal ([1], Proposition 1.2).

In this paper, we propose a sufficient condition for non-idealness that is purely combinatorial.

Definition 2 \mathcal{C} is k -wise intersecting if every subset of at most k members have a common element, yet no element belongs to all members.

Note that for $k = 2$, this notion coincides with the notion of intersecting clutters. Furthermore, for $k \geq 3$, a k -wise intersecting clutter is also $(k-1)$ -wise intersecting. The following is our main conjecture.

Conjecture 3 *There exists an integer $k \geq 4$ such that every k -wise intersecting clutter is non-ideal.*

In fact, we conjecture that the above holds for $k = 4$. We prove this for an important class of clutters. A clutter is *binary* if the symmetric difference of any odd number of members contains a member [22]. Many rich classes of clutters are in fact binary [12]. For example, given a graph $G = (V, E)$ and distinct vertices s and t , the clutter of st -paths over ground set E is binary. The clutter Q_6 is also binary. As evidence for Conjecture 3, our main result is that it holds for all binary clutters.

Theorem 4 *Every 4-wise intersecting binary clutter is non-ideal.*

We also show that 4 cannot be replaced by 3 in Conjecture 3, even for binary clutters.

Proposition 5 *There exists an ideal 3-wise intersecting binary clutter.*

The example from Proposition 5 comes from the Petersen graph, and also coincides with the clutter T_{30} from [26], §79.3e. It has 30 elements and is the smallest such example that we are aware of.

1.1 Paper outline

Let us provide a section by section outline of the paper.

§2: As further evidence for Conjecture 3, we also show that it follows for $k = 5$ from an unpublished conjecture by Seymour from 1975 that was documented in [26], §79.3e.

§3: The theory of blocking clutters allows us to give a very attractive reformulation of Conjecture 3 in terms of the chromatic number of a clutter.

§4: We show that a special class of clutters, called *cuboids*, sit at the heart of Conjecture 3. Cuboids allow us to reformulate Conjecture 3 yet again, but this time in terms of set systems.

§5: We prove Theorem 4 and Proposition 5. Besides the theory of cuboids, two other key ingredients of our proof of Theorem 4 are Jaeger’s *8-flow Theorem* [21] for graphs, and Seymour’s characterization of the binary matroids with the *sums of circuits property* [30].

§6: We propose a line of attack for tackling Conjecture 3 via a deep connection to *projective geometries over the two-element field*, objects that give rise to k -wise intersecting clutters.

§7: We discuss two applications of Theorem 4, one to uniform cycle covers in graphs, another to quarter-integral packings of value two in ideal binary clutters.

2 Dyadic fractional packings in ideal clutters

In this section, we prove that Conjecture 3 for $k = 5$ follows from a conjecture of Seymour.

Let \mathcal{C} be a clutter over ground set V . A *fractional packing of \mathcal{C}* is a vector $y \in \mathbb{R}_+^{\mathcal{C}}$ such that $\sum (y_C : C \in \mathcal{C}, v \in C) \leq 1$ for all $v \in V$; its *value* is $\mathbf{1}^\top y$. Observe that a fractional packing is nothing but a feasible solution to (D), below.

$$\begin{aligned} (P) \quad & \min \quad \mathbf{1}^\top x \\ & \text{s.t.} \quad \sum (x_u : u \in C) \geq 1 \quad \forall C \in \mathcal{C} \\ & \quad \quad x \geq \mathbf{0} \\ (D) \quad & \max \quad \mathbf{1}^\top y \\ & \text{s.t.} \quad \sum (y_C : u \in C \in \mathcal{C}) \leq 1 \quad \forall u \in V \\ & \quad \quad y \geq \mathbf{0}. \end{aligned}$$

As a result, Weak LP Duality implies that every fractional packing has value at most $\tau(\mathcal{C})$. Moreover, if \mathcal{C} is an ideal clutter, then Strong LP Duality guarantees the existence of a fractional packing of value $\tau(\mathcal{C})$. Seymour conjectures the following.

Conjecture 6 (Seymour 1975, see [26], §79.3e) *Every ideal clutter \mathcal{C} has a $\frac{1}{4}$ -integral packing of value $\tau(\mathcal{C})$.*

Here, a vector is $\frac{1}{4}$ -integral if every entry is $\frac{1}{4}$ -integral.

Let us prove that Conjecture 6 implies Conjecture 3. Let I and J be disjoint subsets of V . The *minor $\mathcal{C} \setminus I/J$* obtained after *deleting I* and *contracting J* is the clutter over ground set $V - (I \cup J)$ whose members are the minimal sets in $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. If $J = \emptyset$, then $\mathcal{C} \setminus I/J = \mathcal{C} \setminus I$ is called a *deletion minor*. If a clutter is ideal, then so is every minor of it [28].

Proposition 7 *If Conjecture 6 is true, then Conjecture 3 is true for $k = 5$.*

Proof Assume Conjecture 6 is true. Let \mathcal{C} be an ideal clutter with $\tau(\mathcal{C}) \geq 2$. We need to exhibit at most five members without a common element. By Conjecture 6, \mathcal{C} has a $\frac{1}{4}$ -integral packing $y \in \mathbb{R}_+^{\mathcal{C}}$ of value $\tau(\mathcal{C}) \geq 2$. Pick a minimal subset $\mathcal{C}' \subseteq \{\mathcal{C} \in \mathcal{C} : y_{\mathcal{C}} > 0\}$ such that $\sum_{\mathcal{C} \in \mathcal{C}'} y_{\mathcal{C}} > 1$. Clearly, $|\mathcal{C}'| \leq 5$. Moreover, the members of \mathcal{C}' do not have a common element; for if u was a common element to the members of \mathcal{C}' , then

$$1 < \sum_{\mathcal{C} \in \mathcal{C}'} y_{\mathcal{C}} = \sum_{u \in \mathcal{C} \in \mathcal{C}'} y_{\mathcal{C}} = \sum_{u \in \mathcal{C} \in \mathcal{C}} y_{\mathcal{C}} \leq 1$$

where the last inequality holds because y is a fractional packing, a contradiction. As a result, \mathcal{C}' , and therefore \mathcal{C} , has a subset of at most five members without a common element, as required. \square

Conjecture 6 is notoriously difficult as it is open even for binary clutters. In fact, it remains open for the clutter of *postman sets* of a graph ([12], Conjecture 2.15). Proposition 7 is particularly interesting as it suggests a clue for proving Conjecture 6, by tackling Conjecture 3 first. As we see in §7, Theorem 4 implies Conjecture 6 for ideal binary clutters with covering number two.

3 The chromatic number of a clutter

In this section, we give a very attractive reformulation of Conjecture 3.

Let \mathcal{C} be a clutter over ground set V where every member has cardinality at least two. For an integer $k \geq 2$, a *proper k -colouring* of \mathcal{C} is an assignment of k colours to the elements V such that no monochromatic member exists, that is, it is a partition of V into k parts none of which contains a member.

Definition 8 *The chromatic number of \mathcal{C} , denoted $\chi(\mathcal{C})$, is the smallest integer k such that a proper k -colouring exists.*

Since every member of \mathcal{C} has cardinality at least two, $\chi(\mathcal{C})$ is well-defined.

The *incidence matrix* of \mathcal{C} , denoted $M(\mathcal{C})$, is the matrix whose columns are indexed by the elements and whose rows are the incidence vectors of the members. We say that \mathcal{C} is a *balanced* clutter if $M(\mathcal{C})$ does not have the adjacency matrix of an odd circuit as a submatrix. Balanced clutters form an important subclass of ideal clutters. If \mathcal{C} is a balanced clutter where every member has cardinality at least two, then $\chi(\mathcal{C}) \leq 2$ (and therefore $= 2$). Balancedness, as well as the statements just made, are due to Claude Berge [8].

Given Berge's result in our context, one may wonder whether a universal upper bound exists for the chromatic number of an ideal clutter? We conjecture the following extension.

Conjecture 9 *There exists an integer $k \geq 4$ such that every ideal clutter without a member of cardinality at most one has chromatic number at most k .*

We prove that this conjecture is, in fact, equivalent to Conjecture 3! To show this, we need to introduce an important clutter notion.

Let \mathcal{C} be a clutter over ground set V . The family of minimal covers of \mathcal{C} forms another clutter over the ground set V , called the *blocker* of \mathcal{C} and denoted $b(\mathcal{C})$. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [20, 15].

Proposition 10 *Let \mathcal{C} be a clutter over ground set V where every member has cardinality at least two. Let $k \geq 2$ be an integer. Then \mathcal{C} has chromatic number at most k if, and only if, $b(\mathcal{C})$ is not k -wise intersecting.*

Proof (\Rightarrow) Let A_1, \dots, A_k be a proper k -colouring of \mathcal{C} , that is, A_1, \dots, A_k form a partition of V such that no $A_i, i \in [k] := \{1, \dots, k\}$ contains a member of \mathcal{C} . Then for each $i \in [k]$, $V - A_i$ is a cover of \mathcal{C} , so there exists $B_i \in b(\mathcal{C})$ disjoint from A_i . Since $A_1 \cup \dots \cup A_k = V$, it follows that $B_1 \cap \dots \cap B_k = \emptyset$, certifying that $b(\mathcal{C})$ is not k -wise intersecting.

(\Leftarrow) Assume that $b(\mathcal{C})$ is not k -wise intersecting. Since every member of \mathcal{C} has cardinality at least two, $\tau(b(\mathcal{C})) \geq 2$, so there exist $B_1, \dots, B_k \in b(\mathcal{C})$ such that $B_1 \cap \dots \cap B_k = \emptyset$. Let $A_1 := V - B_1$ and for $i = 2, \dots, k$ let

$$A_i := (B_1 \cap \dots \cap B_{i-1}) - B_i.$$

By definition, A_1, \dots, A_k are pairwise disjoint. As $B_1 \cap \dots \cap B_k = \emptyset$, it follows that A_1, \dots, A_k partition V . As no A_i is a cover of $b(\mathcal{C})$, it follows that A_1, \dots, A_k is a proper k -colouring of $b(\mathcal{C}) = \mathcal{C}$, so \mathcal{C} has chromatic number at most k . \square

A clutter is ideal if, and only if, its blocker is ideal [17, 23, 12]. We are now ready to prove the following.

Proposition 11 *Conjecture 3 for k is equivalent to Conjecture 9 for the same k .*

Proof Assume that Conjecture 3 holds for k , that is, every k -wise intersecting clutter is non-ideal. Let \mathcal{C} be an ideal clutter where every member has cardinality at least two. Then $b(\mathcal{C})$ is an ideal clutter, so it is not k -wise intersecting by our hypothesis, so $\chi(\mathcal{C}) \leq k$ by Proposition 10. Thus, Conjecture 9 holds for the same k . The other direction is similar. \square

A result that must be noted in the context of Conjecture 9 is the following: For every ideal clutter \mathcal{C} where every member and also every cover has cardinality at least two, either $\chi(\mathcal{C}) \leq 2$ or $\chi(b(\mathcal{C})) \leq 2$ (or both). This result was proved recently, and the proof relied on *Gauge Duality in Quadratic Programming* [3].

As a final note, it is well-known that a clutter is binary if, and only if, the blocker is binary [22]. As a consequence, Theorem 4 and Proposition 5 (which have not yet been proved) are equivalent, respectively, to the following statements.

Theorem 12 *Every ideal binary clutter without a member of cardinality at most one has chromatic number at most 4.*

Proposition 13 *There exists an ideal binary clutter whose chromatic number is equal to 4.*

4 Cuboids

In this section, we reformulate Conjecture 3 in terms of cuboids. The concept of cuboids is key for the proof of Theorem 4.

Let $n \geq 1$ be an integer, and let $S \subseteq \{0, 1\}^n$. The *cuboid of S* , denoted $\text{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors $(p_1, 1 - p_1, \dots, p_n, 1 - p_n)$ over all $(p_1, \dots, p_n) \in S$. We say that a clutter is a *cuboid* if, after a relabeling of its ground set, is equal to $\text{cuboid}(S)$, for some S .

Observe that for each $C \in \text{cuboid}(S)$, $|C \cap \{2i - 1, 2i\}| = 1$ for all $i \in [n]$. In particular, every member of $\text{cuboid}(S)$ has size n (hence $\text{cuboid}(S)$ is a clutter) and $\tau(\text{cuboid}(S)) \leq 2$. Cuboids were introduced in [6] and further studied in [1].

We now describe what it means for $\text{cuboid}(S)$ to be k -wise intersecting. We say that the points in S *agree on a coordinate* if $S \subseteq \{x : x_i = a\}$ for some coordinate $i \in [n]$ and some $a \in \{0, 1\}$.

Remark 14 *Let $S \subseteq \{0, 1\}^n$. Then $\text{cuboid}(S)$ is a k -wise intersecting clutter if, and only if, the points in S do not agree on a coordinate yet every subset of at most k points do.*

Next, we describe what it means for $\text{cuboid}(S)$ to be ideal. Let $\text{conv}(S)$ denote the convex hull of S . An inequality of the form $\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1$, for some disjoint $I, J \subseteq [n]$, is called a *generalized set covering inequality* [12]. The set S is *cube-ideal* if every facet of $\text{conv}(S)$ is defined by $x_i \geq 0$, $x_i \leq 1$, or a generalized set covering inequality [1].

Theorem 15 ([1]) *Let $S \subseteq \{0, 1\}^n$. Then $\text{cuboid}(S)$ is an ideal clutter if, and only if, S is a cube-ideal set.*

As a result, (the contrapositive of) Conjecture 3 for cuboids reduces to the following conjecture.

Conjecture 16 *There exists an integer $k \geq 4$ such that for every cube-ideal set, either all the points agree on a coordinate, or there is a subset of at most k points that do not agree on a coordinate.*

Surprisingly, we show that Conjecture 16 is equivalent to Conjecture 3! We need the notions of a *tangled* clutter and the *core* of an ideal tangled clutter.

4.1 Tangled clutters

Take an integer $k \geq 2$. If every k members of a clutter have a common element, then so do every k members of a deletion minor. Furthermore, for every element $v \in V$, $\tau(\mathcal{C}) \geq \tau(\mathcal{C} \setminus v) \geq \tau(\mathcal{C}) - 1$, where $\tau(\mathcal{C} \setminus v) = \tau(\mathcal{C}) - 1$ if and only if v belongs to some minimum cover of \mathcal{C} . Motivated by these observations, we make the following definition.

Definition 17 \mathcal{C} is a tangled clutter if $\tau(\mathcal{C}) = 2$ and every element belongs to a minimum cover.

Observe that every cuboid with covering number two is a tangled clutter. Tangled clutters are relevant due to the following remark.

Remark 18 Let \mathcal{C} be a k -wise intersecting clutter. Let \mathcal{C}' be a deletion minor of \mathcal{C} that is minimal subject to $\tau(\mathcal{C}') \geq 2$. Then \mathcal{C}' is a tangled k -wise intersecting clutter.

Thus, it suffices to prove Conjecture 3 for tangled clutters. In the rest of this section, we further reduce the conjecture to cuboids.

4.2 Every ideal tangled clutter has an ideal core.

Let \mathcal{C} be an ideal tangled clutter over ground set V . Consider the following dual pair of linear programs:

$$\begin{aligned} (P) \quad & \min \quad \mathbf{1}^\top x \\ & \text{s.t.} \quad \sum (x_u : u \in C) \geq 1 \quad \forall C \in \mathcal{C} \\ & \quad \quad x \geq \mathbf{0} \\ (D) \quad & \max \quad \mathbf{1}^\top y \\ & \text{s.t.} \quad \sum (y_C : u \in C \in \mathcal{C}) \leq 1 \quad \forall u \in V \\ & \quad \quad y \geq \mathbf{0} \end{aligned}$$

As \mathcal{C} has covering number two, (P) has optimal value two, and every minimum cover of \mathcal{C} yields an optimum to (P). By Strong LP Duality, the optimal value of (D) is also two; let y^* be a fractional packing of value two. By Complementary Slackness, whenever $y_C^* > 0$, then $|C \cap \{u, v\}| = 1$ for every minimum cover $\{u, v\}$ of \mathcal{C} . This observation motivates the following definition.

Definition 19 Let \mathcal{C} be an ideal tangled clutter over ground set V . The core of \mathcal{C} is the clutter

$$\text{core}(\mathcal{C}) := \{C \in \mathcal{C} : |C \cap \{u, v\}| = 1 \text{ for every minimum cover } \{u, v\}\}.$$

In this subsection, we prove that the core of every ideal tangled clutter is another ideal tangled clutter, one that arises from a cuboid. We need a few ingredients.

Let \mathcal{C} be a clutter over ground set V . Let

$$Q(\mathcal{C}) := \left\{ x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \right\}.$$

Observe that the 0–1 points in $Q(\mathcal{C})$ are precisely the incidence vectors of the covers of \mathcal{C} , while the integral vertices of $Q(\mathcal{C})$ are precisely the incidence vectors of the minimal covers of \mathcal{C} . Basic polyhedral theory tells us that \mathcal{C} is

an ideal clutter if, and only if, $Q(\mathcal{C})$ is another integral polyhedron (see [11], Theorem 4.1).

To *duplicate an element* u of \mathcal{C} is to introduce a new element \bar{u} , and replace \mathcal{C} by the clutter over ground set $V \cup \{\bar{u}\}$, whose members are $\{C : C \in \mathcal{C}, u \notin C\} \cup \{C \cup \{\bar{u}\} : C \in \mathcal{C}, u \in C\}$. A *duplication* of \mathcal{C} is a clutter obtained from \mathcal{C} by repeatedly duplicating elements. We leave it as an exercise for the reader to check that a clutter is ideal if and only if some duplication of it is ideal. It can also be easily checked that a clutter is k -wise intersecting if and only if some duplication of it is k -wise intersecting.

Proposition 20 *Let \mathcal{C} be an ideal tangled clutter over ground set V . Let $G = (V, E)$ be the graph whose edges correspond to the minimum covers of \mathcal{C} . Then the following statements hold:*

1. $\text{core}(\mathcal{C}) = \{C \in \mathcal{C} : y_C > 0 \text{ for some fractional packing } y \text{ of value two}\}$,
2. G is a bipartite graph, and
3. if $\{U, U'\}$ is the bipartition of a connected component of G , then the elements in U (resp. U') are duplicates in $\text{core}(\mathcal{C})$, and $|\{u, u'\} \cap C| = 1$ for all $u \in U, u' \in U'$ and $C \in \text{core}(\mathcal{C})$.

Proof (1) (\supseteq) follows from Complementary Slackness while (\subseteq) follows from Strict Complementarity. (2) It follows from (1) that $\text{core}(\mathcal{C}) \neq \emptyset$. It can now be readily checked that G is a bipartite graph, as for each $C \in \text{core}(\mathcal{C})$, $\{C, V - C\}$ is a bipartition of G . (3) If $\{u, v\}, \{u, w\}$ are minimum covers of \mathcal{C} , then v, w are duplicates in $\text{core}(\mathcal{C})$. This observation proves (3). \square

We need the following lemma.

Lemma 21 ([6], Lemma 3.1) *Let \mathcal{C} be a clutter whose ground set can be partitioned into parts $\{u_1, v_1\}, \dots, \{u_r, v_r\}$ such that $|\{u_i, v_i\} \cap C| = 1$ for each $i \in [r]$ and $C \in \mathcal{C}$. Then \mathcal{C} is ideal if, and only if, $\text{conv}\{\chi_C : C \in \mathcal{C}\} = Q(b(\mathcal{C})) \cap \{x : x_{u_i} + x_{v_i} = 1 \forall i \in [r]\}$.*

Here, χ_C denotes the incidence vector of C .

We are now ready to prove the main result of this subsection.

Theorem 22 *Let \mathcal{C} be an ideal tangled clutter. Then $\text{core}(\mathcal{C})$ is a duplication of a cuboid. Moreover, $\text{core}(\mathcal{C})$ is an ideal tangled clutter.*

Proof Denote by V the ground set of \mathcal{C} . Let $G = (V, E)$ be the graph whose edges correspond to the minimum covers of \mathcal{C} . By Proposition 20 (2), G is a bipartite graph. Let r be the number of connected components of G , and for each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component. By Proposition 20 (3), for each $i \in [r]$, the elements in U_i are duplicates in $\text{core}(\mathcal{C})$, the elements in V_i are duplicates in $\text{core}(\mathcal{C})$, and $|\{u, v\} \cap C| = 1$ for all $u \in U_i, v \in V_i$ and $C \in \text{core}(\mathcal{C})$. That is, each $C \in \text{core}(\mathcal{C})$ is determined by r binary choices; in each connected component of G , C must contain exactly one

of the two parts of the bipartition. This allows a more concise representation of the core. For each $C \in \text{core}(\mathcal{C})$, define $p_C \in \{0, 1\}^r$ such that

$$(p_C)_i = \begin{cases} 0 & \text{if } C \cap (U_i \cup V_i) = V_i \\ 1 & \text{if } C \cap (U_i \cup V_i) = U_i \end{cases}$$

Let $S := \{p_C : C \in \text{core}(\mathcal{C})\} \subseteq \{0, 1\}^r$. Then $\text{core}(\mathcal{C})$ is a duplication of cuboid(S).

Claim 1 $\text{core}(\mathcal{C})$ is a tangled clutter.

Proof (of Claim) As a subset of \mathcal{C} , $\text{core}(\mathcal{C})$ has covering number at most two, and every element of it appears in a cover of cardinality two. Thus, to prove the claim, it suffices to show that $\text{core}(\mathcal{C})$ has covering number at least two. Let y be a fractional packing of \mathcal{C} of value two. Then $\text{support}(y) \subseteq \text{core}(\mathcal{C})$ by Proposition 20 (1), so y is also a fractional packing of $\text{core}(\mathcal{C})$. Subsequently, $\text{core}(\mathcal{C})$ has covering number at least two, as required. \diamond

It remains to prove that $\text{core}(\mathcal{C})$ is an ideal clutter; we use Lemma 21 to prove this. We know that

$$\{\chi_C : C \in \text{core}(\mathcal{C})\} = \{\chi_C : C \in \mathcal{C}\} \cap \{x : x_u + x_v = 1, \{u, v\} \in E\}. \quad (\star)$$

Claim 2 $\text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\} = Q(b(\text{core}(\mathcal{C}))) \cap \{x : x_u + x_v = 1, \{u, v\} \in E\}$.

Proof (of Claim) (\subseteq) follows immediately from (\star) . (\supseteq) Pick a point x^* in the set on the right-hand side. As $Q(b(\text{core}(\mathcal{C}))) \subseteq Q(b(\mathcal{C}))$, we have $x^* \in Q(b(\mathcal{C}))$. Since \mathcal{C} is ideal, so is $b(\mathcal{C})$, implying that for some $\lambda \in \mathbb{R}_+^{\mathcal{C}}$ with $\sum_{C \in \mathcal{C}} \lambda_C = 1$, we have that

$$x^* \geq \sum_{C \in \mathcal{C}} \lambda_C \chi_C.$$

Since for all $\{u, v\} \in E$, we have that $x_u^* + x_v^* = 1$ and $\{u, v\} \in b(\mathcal{C})$, equality must hold above and by (\star) , if $\lambda_C > 0$ then $C \in \text{core}(\mathcal{C})$. Hence, $x^* \in \text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\}$, as required. \diamond

For each $i \in [r]$, pick $u_i \in U_i$ and $v_i \in V_i$, and let \mathcal{C}' be the clutter over ground set $\{u_1, v_1, \dots, u_r, v_r\}$ obtained from $\text{core}(\mathcal{C})$ after contracting $V - \{u_1, v_1, \dots, u_r, v_r\}$. Notice that $|\{u_i, v_i\} \cap C| = 1$ for all $i \in [r]$ and $C \in \mathcal{C}'$. (Observe that \mathcal{C}' is nothing but cuboid(S).)

Claim 3 $\text{conv}\{\chi_C : C \in \mathcal{C}'\} = Q(b(\mathcal{C}')) \cap \{z : z_{u_i} + z_{v_i} = 1 \ i \in [r]\}$.

Proof (of Claim) We use Claim 2 to prove this equality. Observe that $\text{conv}\{\chi_C : C \in \mathcal{C}'\}$ is the projection of $\text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\}$ onto the coordinates $\{u_i, v_i : i \in [r]\}$. Thus, to finish the proof, it suffices to show that $Q(b(\mathcal{C}')) \cap \{z : z_{u_i} + z_{v_i} = 1, i \in [r]\}$ is the projection of $Q(b(\text{core}(\mathcal{C}))) \cap \{x : x_u + x_v = 1, \{u, v\} \in E\}$ onto the same coordinates. We leave this as an easy exercise for the reader. \diamond

It therefore follows from Lemma 21 that \mathcal{C}' is an ideal clutter. As $\text{core}(\mathcal{C})$ is a duplication of \mathcal{C}' , $\text{core}(\mathcal{C})$ is an ideal clutter, too, thereby completing the proof. \square

4.3 Conjectures 16 and 3 are equivalent.

Theorem 23 *Conjecture 16 for k is equivalent to Conjecture 3 for the same k .*

Proof We already showed (\Leftarrow) . It remains to prove (\Rightarrow) . Suppose Conjecture 3 is false for some $k \geq 4$. That is, there is an ideal k -wise intersecting clutter \mathcal{C} . Let \mathcal{C}' be a deletion minor of \mathcal{C} that is minimal subject to $\tau(\mathcal{C}') \geq 2$. By Remark 18, \mathcal{C}' is an ideal tangled k -wise intersecting clutter. By Theorem 22, $\text{core}(\mathcal{C}')$ is an ideal tangled clutter that is a duplication of some cuboid, say $\text{cuboid}(S)$. As every k members of \mathcal{C}' have a common element, so do every k members of $\text{core}(\mathcal{C}')$, so the latter is k -wise intersecting. As a result, $\text{cuboid}(S)$ is an ideal k -wise intersecting clutter, so by Remark 14 and Theorem 15, S is a cube-ideal set whose points do not agree on a coordinate yet every subset of $\leq k$ points do. Therefore, S refutes Conjecture 16 for k , as required. \square

5 Graphs, binary matroids, and binary clutters

In this section, we prove Theorem 4 and Proposition 5. More specifically, in §5.1, we explain how every graph leads to a cube-ideal set and how the 8-Flow Theorem proves Conjecture 16 for such cube-ideal sets; we also give an example of an ideal 3-wise intersecting binary clutter, thereby proving Proposition 5. After a primer on binary matroids in §5.2, we introduce the class of binary matroids with the sums of circuits property 5.3. Using the machinery developed in earlier subsections, we finally prove Theorem 4 in §5.4.

A key notion throughout this section is the following. Let $S \subseteq \{0, 1\}^n$. For $x, y \in \{0, 1\}^n$, $x \Delta y$ denotes the coordinate-wise sum of x, y modulo 2. We say that S is a *vector space over $GF(2)$* , or simply a *binary space*, if $a \Delta b \in S$ for all $a, b \in S$. Notice that a nonempty binary space necessarily contains $\mathbf{0}$.

Remark 24 *Let $S \subseteq \{0, 1\}^n$ contain $\mathbf{0}$. Then $\text{cuboid}(S)$ is a binary clutter if, and only if, S is a binary space.*

Recall that a clutter is binary if the symmetric difference of any odd number of members contains a member. A binary clutter can be characterized as the *port* of a binary matroid, and also as the clutter of the *odd circuits* of a *signed binary matroid* (see [19]). The above remark shows yet another way to obtain a binary clutter from a binary matroid.

5.1 The 8-Flow Theorem

Let $G = (V, E)$ be a graph where loops and parallel edges are allowed, where every loop is treated as an edge not incident to any vertex. A *cycle* is a subset $C \subseteq E$ such that every vertex is incident with an even number of edges in C . A *bridge* of G is an edge e that does not belong to any cycle. The *cycle space* of G is the set

$$\text{cycle}(G) := \{\chi_C : C \subseteq E \text{ is a cycle}\} \subseteq \{0, 1\}^E.$$

As \emptyset is a cycle, and the symmetric difference of any two cycles is also a cycle, it follows that $\text{cycle}(G)$ is a binary space. We require the following two results on cycle spaces of graphs.

Remark 25 *Let $G = (V, E)$ be a graph. Then the points in $\text{cycle}(G)$ agree on a coordinate if, and only if, G has a bridge. Moreover, for all $k \in \mathbb{N}$, $\text{cycle}(G)$ has a subset of at most $k + 1$ points that do not agree on a coordinate if, and only if, G has at most k cycles the union of which is E .*

Theorem 26 ([29, 7], see [1]) *The cycle space of every graph is a cube-ideal set.*

We need the following version of the celebrated 8-Flow Theorem of Jaeger [21]. (The second equivalent statement follows from Remark 25.)

Theorem 27 ([21]) *Let $G = (V, E)$ be a graph. Then either G has a bridge, or there are most 3 cycles the union of which is E . That is, given $\text{cycle}(G) \subseteq \{0, 1\}^E$, either all the points agree on a coordinate, or there is a subset of at most 4 points that do not agree on a coordinate.*

One may wonder whether the 3, 4 in Theorem 27 may be replaced by 2, 3? The answer is no, due to the Petersen graph (see Figure 1a):

Remark 28 (see [31]) *The edge set of the Petersen graph is not the union of 2 cycles.*

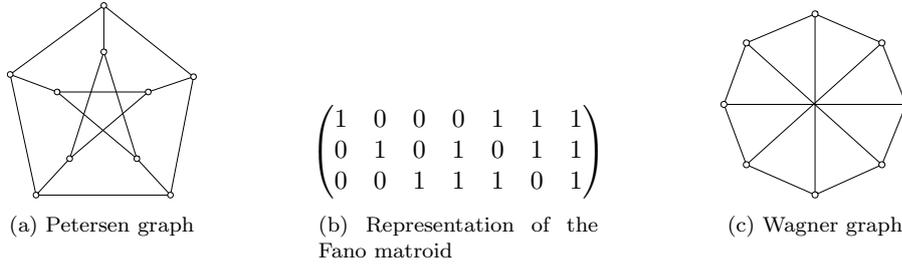


Fig. 1

As a consequence, an ideal 3-wise intersecting clutter does exist:

Proof (of Proposition 5) Let S be the cycle space of the Petersen graph, and let $\mathcal{C} := \text{cuboid}(S)$. By Remark 24, \mathcal{C} is a binary clutter. By Remark 28, the Petersen is a bridgeless graph that does not have 2 cycles the union of which is the edge set, so by Remark 25, the points in S do not agree on a coordinate, but every subset of $2 + 1 = 3$ points do. Moreover, S is a cube-ideal set by Theorem 26. Therefore, by Remark 14 and Theorem 15, \mathcal{C} is an ideal 3-wise intersecting binary clutter, as required. \square

The cuboid of the cycle space of the Petersen graph has already shown up in the literature, and is denoted T_{30} by Schrijver [26], §79.3e. Consider the graph obtained from Petersen by subdividing every edge once, and let T be any vertex subset of even cardinality containing all the new vertices. Then the clutter of *minimal T -joins* of this graph is precisely T_{30} . This construction is due to Seymour ([29], page 440).

5.2 A primer on binary matroids

We follow Oxley [25]. Let E be a finite set, $S \subseteq \{0, 1\}^E$ a binary space, and S^\perp the orthogonal complement of S , that is, $S^\perp = \{y \in \{0, 1\}^E : y^\top x \equiv 0 \pmod{2} \forall x \in S\}$. Notice that S^\perp is another binary space, and that $(S^\perp)^\perp = S$. Therefore, there exists a 0–1 matrix A whose columns are labeled by E such that $S = \{x \in \{0, 1\}^E : Ax \equiv \mathbf{0} \pmod{2}\}$, and S^\perp is the row space of A generated over $GF(2)$.

Let $\mathcal{S} := \{C \subseteq E : \chi_C \in S\}$. The pair $M := (E, \mathcal{S})$ is a *binary matroid*, and the matrix A is a *representation of M* . We call E the *ground set of M* , and denote it by EM . The sets in \mathcal{S} are the *cycles of M* , and \mathcal{S} is the *cycle space of M* , denoted by $\text{cycle}(M)$. The minimal nonempty sets in \mathcal{S} are the *circuits of M* , and the circuits of cardinality one are *loops*.

Let $\mathcal{S}^\perp := \{D \subseteq E : \chi_D \in S^\perp\}$. The binary matroid $M^* := (E, \mathcal{S}^\perp)$ is the *dual of M* . Notice that $(M^*)^* = M$. The sets in \mathcal{S}^\perp are the *cocycles of M* , and \mathcal{S}^\perp is the *cocycle space of M* , denoted by $\text{cocycle}(M)$. The minimal nonempty sets in \mathcal{S}^\perp are the *cocircuits of M* , and the cocircuits of cardinality one are *coloops of M* .

Remark 29 *Let M be a binary matroid. Then the points in $\text{cycle}(M)$ agree on a coordinate if, and only if, M has a coloop. Moreover, for all $k \in \mathbb{N}$, $\text{cycle}(M)$ has a subset of at most $k + 1$ points that do not agree on a coordinate if, and only if, M has at most k cycles the union of which is EM .*

Let $G = (V, E)$ be a graph. The binary matroid whose cycle space is $\text{cycle}(G)$ is a *graphic matroid*, and is denoted $M(G)$. Notice the one-to-one correspondence between the cycles of $M(G)$ and the cycles of G , between the loops of $M(G)$ and the loops of G , between the cocycles of $M(G)$ and the cuts of G , and between the coloops of $M(G)$ and the bridges of G . Therefore, Remark 29 is an extension of Remark 25 to all binary matroids.

Finally, let M be a binary matroid. A pair of distinct elements $e, f \in EM$ are *parallel* if $\{e, f\}$ is a circuit of M . Given distinct elements $e, f, g \in EM$, if e, f are parallel and f, g are parallel, then so are e, g . A *parallel class* of M is a maximal subset of EM of pairwise parallel elements. The *simplification of M* , denoted $\text{si}(M)$, is the binary matroid obtained from M after deleting all loops, and for every parallel class, keeping one representative and deleting all the other elements. By construction, $\text{si}(M)$ is a *simple* binary matroid, i.e. it has no circuit of cardinality at most two.

5.3 The sums of circuits property

A binary matroid M over ground set E has the *sums of circuits property* if for each $w \in \mathbb{R}_+^E$ satisfying

$$w(D - \{f\}) \geq w_f \quad \text{for every cocycle } D \text{ and } f \in D,$$

there exists an assignment $y_C \geq 0$ to every circuit C such that

$$w = \sum (y_C \cdot \chi_C : C \text{ is a circuit}).$$

This notion was first introduced by Seymour where he proved that graphic matroids have the sums of circuits property [29]. This matroid theoretic notion is relevant as it coincides with the notion of cube-idealness for binary spaces.

Theorem 30 ([7, 1]) *A binary matroid has the sums of circuits property if, and only if, the corresponding cycle space is a cube-ideal set.*

Seymour proved a decomposition theorem for binary matroids with the sums of circuits property [30]. It turns out they can all be produced from graphic matroids and two other matroids, which we now describe. The *Fano matroid* F_7 is the binary matroid represented by the matrix in Figure 1b. The second matroid is $M(V_8)^*$, where V_8 is the graph in Figure 1c. Seymour showed that F_7 and $M(V_8)^*$ both have the sums of circuits property [30].

To generate all binary matroids with the sums of circuits property, we require three composition rules. Let M_1, M_2 be binary matroids over ground sets E_1, E_2 , respectively. We denote by $M_1 \Delta M_2$ the binary matroid over ground set $E_1 \Delta E_2$ whose cycles are all subsets of $E_1 \Delta E_2$ of the form $C_1 \Delta C_2$, where C_i is a cycle of M_i for $i \in [2]$. Then $M_1 \Delta M_2$ is a *1-sum* if $E_1 \cap E_2 = \emptyset$; $M_1 \Delta M_2$ is a *2-sum* if $E_1 \cap E_2 = \{e\}$, where e is neither a loop nor a coloop of M_1 or M_2 ; and $M_1 \Delta M_2$ is a *Y-sum* if $E_1 \cap E_2$ is a cocircuit of cardinality 3 in both M_1 and M_2 and contains no circuit in M_1 or M_2 .

Theorem 31 ([30], (6.4), (6.7), (6.10) and (16.4)) *Let M be a binary matroid with the sums of circuits property. Then M is obtained recursively by means of 1-sums, 2-sums and Y-sums starting from copies of $F_7, M(V_8)^*$ and graphic matroids.*

We are now ready to extend Theorem 27 to all binary matroids with the sums of circuits property, as follows. (The second equivalent statement follows from Remark 29 and Theorem 30.)

Theorem 32 *Every binary matroid without a coloop and with the sums of circuits property has at most 3 cycles the union of which is the ground set. That is, given a cube-ideal binary space, either all the points agree on a coordinate, or there is a subset of at most 4 points that do not agree on a coordinate.*

Proof A *3-cycle cover* of a binary matroid is 3 (not necessarily distinct) cycles whose union is the ground set.

Claim 1 *Both F_7 and $M(V_8)^*$ have 3-cycle covers.*

Proof (of Claim) Given the matrix representation of F_7 in Figure 1a, label the columns $1, \dots, 7$ from left to right. Then $\emptyset, \{1, 2, 3, 7\}, \{4, 5, 6\}$ is a 3-cycle cover of F_7 . Next, label the vertices of V_8 so that the outer 8-cycle is labelled $1, \dots, 8$. Then $M(V_8)^*$ has a 3-cycle cover given by the following cuts of V_8 : $\delta(\{1, 6, 7, 8\}), \delta(\{1, 7\}), \delta(\{2, 4\})$, where $\delta(X)$ is the set of edges with exactly one end in X . \diamond

Claim 2 *Let M, M_1, M_2 be binary matroids such that $M = M_1 \triangle M_2$ and $M_i, i \in [2]$ has a 3-cycle cover. Then the following statements hold:*

1. *If M is a 1-sum of M_1, M_2 , then M has a 3-cycle cover.*
2. *If M is a 2-sum of M_1, M_2 , then M has a 3-cycle cover.*
3. *If M is a Y -sum of M_1, M_2 , then M has a 3-cycle cover.*

Proof (of Claim) For $i \in [2]$, let E_i be the ground set of M_i and C_1^i, C_2^i, C_3^i be a 3-cycle cover of M_i . Clearly, **(1)** holds. For **(2)**, let $E_1 \cap E_2 = \{e\}$. We may assume $e \in C_1^i$ for all $i \in [2]$. By replacing C_2^i by $C_1^i \triangle C_2^i$ if necessary, we may assume $e \notin C_2^i$ for all $i \in [2]$. Similarly, we may assume $e \notin C_3^i$ for $i \in [2]$. But now $\{C_j^1 \triangle C_j^2 : j \in [3]\}$ is a 3-cycle cover of M . For **(3)**, suppose $E_1 \cap E_2 = \{e, f, g\}$. Since $\{e, f, g\}$ is a cocircuit of both M_1, M_2 , and since cocircuits and circuits of a binary matroid have an even number of elements in common, $|C_j^i \cap \{e, f, g\}| \in \{0, 2\}$ for all i, j . Therefore, after possibly relabeling e, f, g simultaneously in M_1 and M_2 , and after possibly relabeling C_1^i, C_2^i, C_3^i for all i , we may assume that

- $C_1^i \cap \{e, f, g\} = \{e, f\}$ for all $i \in [2]$, and
- $C_2^i \cap \{e, f, g\} = \{e, g\}$ or $\{f, g\}$ for all $i \in [2]$.

For $i \in [2]$, after possibly replacing C_2^i with $C_2^i \triangle C_1^i$, we may assume $C_2^i \cap \{e, f, g\} = \{e, g\}$. For $i \in [2]$, after possibly replacing C_3^i with $C_3^i \triangle C_1^i, C_3^i \triangle C_2^i$ or $C_3^i \triangle C_1^i \triangle C_2^i$, we may assume $C_3^i \cap \{e, f, g\} = \emptyset$. But now $\{C_j^1 \triangle C_j^2 : j \in [3]\}$ is a 3-cycle cover of M , as required. \diamond

We leave the proof of the following claim as an easy exercise for the reader.

Claim 3 *Let M, M_1, M_2 be binary matroids such that $M = M_1 \triangle M_2$, where \triangle is either a 1-, 2- or Y -sum. If M has no coloop, then neither do M_1, M_2 . \diamond*

The proof is completed by combining the above claims with Theorems 27 and 31. \square

5.4 Proof of Theorem 4

It is well-known that a clutter is binary if, and only if, every member and every minimal cover have an odd number of elements in common [22]. We use this characterization to prove the following.

Proposition 33 *Let \mathcal{C} be a binary tangled clutter. Then \mathcal{C} is a duplication of a cuboid.*

Proof If $\{e, f\}$ is a minimum cover, then for each $C \in \mathcal{C}$, $|C \cap \{e, f\}|$ must be odd and therefore 1, since \mathcal{C} is a binary clutter. As a result, if $\{e, f\}, \{e, g\}$ are both minimum covers, then f, g must be duplicates. Moreover, if every element is contained in exactly one minimum cover, then \mathcal{C} must be a cuboid. These two observations, along with the fact that \mathcal{C} is a tangled clutter, imply that \mathcal{C} is a duplication of a cuboid. \square

Every minor of a binary clutter is also a binary clutter [27]. We are now equipped to prove the main result of this paper, stating that every 4-wise intersecting binary clutter is non-ideal.

Proof (of Theorem 4) We prove the contrapositive statement. Let \mathcal{C} be an ideal binary clutter such that $\tau(\mathcal{C}) \geq 2$. We need to exhibit ≤ 4 members without a common element. Let \mathcal{C}' be a deletion minor of \mathcal{C} that is minimal subject to $\tau(\mathcal{C}') \geq 2$. It suffices to exhibit ≤ 4 members of \mathcal{C}' without a common element. Notice that \mathcal{C}' is ideal, and as a minor of a binary clutter, it is also binary. Moreover, by our minimality assumption, \mathcal{C}' is a tangled clutter. Thus, by Proposition 33, \mathcal{C}' is a duplication of a cuboid, say $\text{cuboid}(S)$ where we may choose S so that $\mathbf{0} \in S$. It suffices to exhibit ≤ 4 members of $\text{cuboid}(S)$ without a common element.

Note that $\text{cuboid}(S)$ is an ideal binary cuboid with $\tau(\text{cuboid}(S)) \geq 2$. So, by Theorem 15 and Remark 24, S is a cube-ideal binary space whose points do not agree on a coordinate. By Theorem 32, S has ≤ 4 points that do not agree on a coordinate, thereby yielding ≤ 4 members of $\text{cuboid}(S)$ without a common element, as required. \square

6 Projective geometries over the two-element field

In this section, we give an important example of k -wise intersecting clutters, coming from projective geometries over the two-element field. We also propose a strengthening of Conjecture 3. Using Theorem 4, we prove our stronger conjecture for the class of binary clutters.

Conjecture 3 predicts that for some $k \geq 4$, every ideal clutter with covering number at least two has k members without a common element. By moving to a deletion minor, if necessary, we may assume that our ideal clutter is tangled. Roughly speaking, our stronger conjecture predicts that the tangled deletion minor must actually have 2^{k-1} members that come from a projective geometry, and of these members, k many will not have a common element.

Take an integer $\ell \geq 1$. Let A be the $\ell \times (2^\ell - 1)$ matrix whose columns are all the nonzero vectors in $\{0, 1\}^\ell$. The binary matroid represented by A is called a *projective geometry over $GF(2)$* , and is denoted $PG(\ell - 1, 2)$. See Figure 2 for representations of the first three projective geometries. Note that $PG(1, 2)$ is the graphic matroid of a triangle, while $PG(2, 2)$ is the Fano matroid F_7 .

$$(1) \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Fig. 2: Representations of $PG(0, 2)$, $PG(1, 2)$, $PG(2, 2)$, from left to right.

Projective geometries are relevant as they give rise to an important class of ℓ -wise intersecting clutters, as we see below. Given integers $n, m \geq 1$ and some points $a_1, \dots, a_m \in \{0, 1\}^n$, denote by $\langle a_1, \dots, a_m \rangle$ the vector space over $GF(2)$ generated by the points a_1, \dots, a_m .

Proposition 34 *Take an integer $\ell \geq 1$. Then the following statements hold:*

1. $PG(\ell - 1, 2)$ has rank ℓ .
2. $PG(\ell - 1, 2)$ has exactly 2^ℓ cocycles.
3. $PG(\ell - 1, 2)$ has a unique representation, up to permuting rows and columns.
4. Every nonempty cocycle of $PG(\ell - 1, 2)$ has cardinality $2^{\ell-1}$.

Moreover, for $S := \text{cocycle}(PG(\ell - 1, 2))$, the following statements hold:

5. $\text{cuboid}(S)$ has $\ell + 1$ members without a common element.
6. $\text{cuboid}(S)$ is an ℓ -wise intersecting clutter.

Proof Let A be the representation of $PG(\ell - 1, 2)$, i.e. the $\ell \times (2^\ell - 1)$ matrix whose columns are all the nonzero vectors in $\{0, 1\}^\ell$. **(1)** and **(2)** follow from the facts that the rows of A are linearly dependent over $GF(2)$, and that $\text{cocycle}(PG(\ell - 1, 2))$ is the row space of A generated over $GF(2)$. **(3)** follows from **(1)** and the fact that $PG(\ell - 1, 2)$ has $2^\ell - 1$ elements, has no loop, and has no parallel elements. **(4)** Every nonempty cocycle can be viewed as the first row of A , the unique representation of $PG(\ell - 1, 2)$, and since the rows of A have precisely $2^{\ell-1}$ ones, (4) follows. **(5)** Note that the ℓ points in S corresponding to the rows of A agree on precisely one coordinate, which is set to 1. These ℓ points, together with the zero point $\mathbf{0}$, yield $\ell + 1$ points that do not agree on a coordinate. **(6)** Let a_1, \dots, a_ℓ be some points in S . Let r be the rank of $\langle a_1, \dots, a_\ell \rangle$. Clearly, $0 \leq r \leq \ell$. If $r = \ell$, then by (3), the ℓ points may be viewed as the ℓ rows of A , so they agree on precisely one coordinate, which is set to 1. Otherwise, $r < \ell$. We may assume that $\{a_1, \dots, a_r\}$ form a basis for $\langle a_1, \dots, a_\ell \rangle$. Then, by (3), $\{a_1, \dots, a_r\}$ may be viewed as r distinct rows of A , implying that they agree on $2^{\ell-r} - 1$ coordinates set to 0, implying in turn that all the points in $\{a_1, \dots, a_\ell\}$ agree on $2^{\ell-r} - 1 \geq 1$ coordinates set to 0, as required. \square

Observe that $\text{cocycle}(PG(0, 2)) = \{0, 1\}$ and $\text{cocycle}(PG(1, 2)) = \{000, 011, 101, 110\}$. In particular,

$$\text{cuboid}(\text{cocycle}(PG(0, 2))) = \{\{1\}, \{2\}\}$$

$$\text{cuboid}(\text{cocycle}(PG(1, 2))) = \{\{2, 4, 6\}, \{2, 3, 5\}, \{1, 4, 5\}, \{1, 3, 6\}\} = Q_6$$

6.1 Embedding projective geometries

A clutter \mathcal{C} embeds $PG(k-2, 2)$ if some subset of \mathcal{C} is a duplication of the cuboid of $\text{cycloid}(PG(k-2, 2))$.

Remark 35 Let \mathcal{C} be a clutter over ground set V . The following statements hold:

1. \mathcal{C} embeds $PG(0, 2)$ if, and only if, \mathcal{C} has two members that partition V ,
2. \mathcal{C} embeds $PG(1, 2)$ if, and only if, \mathcal{C} has four members of the form $I_2 \cup I_4 \cup I_6, I_2 \cup I_3 \cup I_5, I_1 \cup I_4 \cup I_5, I_1 \cup I_3 \cup I_6$, where I_1, \dots, I_6 are nonempty sets that partition V

Cornuéjols, Guenin and Margot refer to the equivalent condition in Remark 35 (2) as the Q_6 property [13]. In that paper, they prove that a subclass of ideal tangled clutters, namely, ideal *minimally non-packing* clutters with covering number two, enjoys the Q_6 property. We propose the following conjecture extending their result to all ideal tangled clutters.

Conjecture 36 There exists an integer $\ell \geq 3$ such that every ideal tangled clutter embeds one of $PG(0, 2), \dots, PG(\ell-1, 2)$.

Let us prove that this conjecture is a strengthening of our main conjecture.

Proposition 37 If Conjecture 36 holds for ℓ , then Conjecture 3 holds for $k = \ell + 1$.

Proof Assume Conjecture 36 holds for ℓ . Let \mathcal{C} be an ideal clutter with $\tau(\mathcal{C}) \geq 2$. We need to exhibit $\leq \ell + 1$ members without a common element. Let \mathcal{C}' be a deletion minor of \mathcal{C} that is minimal subject to $\tau(\mathcal{C}') \geq 2$. Then \mathcal{C}' is an ideal tangled clutter. Thus, \mathcal{C}' embeds $PG(n-1, 2)$ for some $n \in [\ell]$. That is, a duplication of cuboid($PG(n-1, 2)$) is a subset of \mathcal{C}' . By Proposition 34 (5), cuboid($PG(n-1, 2)$) has $n+1$ members without a common element, so the duplication, and therefore \mathcal{C}' , must have $n+1$ members without a common element. Thus, \mathcal{C} has $n+1 \leq \ell+1$ members without a common element, as required. \square

6.2 Conjecture 36 is true for binary clutters.

Proposition 38 Let M be a simple binary matroid where every two elements appear together in a triangle. Then M is a projective geometry.

Proof We may assume that $EM = [r]$ for some integer $r \geq 1$. Let A be a $0-1$ matrix with column labels $[r]$ and whose rows are linearly independent over $GF(2)$, where

$$\text{cycle}(M) = \{x : Ax \equiv \mathbf{0} \pmod{2}\}.$$

After elementary row operations over $GF(2)$ and reordering the columns, if necessary, we may assume that $A = (I \mid A')$ where I is the identity matrix of

appropriate dimension. As M is simple, it follows that every column of A' has at least two 1s, and no two columns of it are equal. Since every two elements of M appear together in a triangle, it follows that if a, b are distinct columns of A , then $a + b \pmod{2}$ is another column. It can now be readily checked that A' consists of all $0-1$ vectors with at least two 1s, implying in turn that M is a projective geometry, as required. \square

Given an integer $n \geq 2$ and a set $S \subseteq \{0,1\}^n$, two distinct coordinates $i, j \in [n]$ are *duplicates in S* if $S \subseteq \{x : x_i = x_j\}$ or $S \subseteq \{x : x_i + x_j = 1\}$. Observe that S has duplicated coordinates if, and only if, $\text{cuboid}(S)$ has duplicated elements. We say that S is a *duplication of S'* if S' is obtained from S after projecting away some duplicated coordinates.

Proposition 39 *Take an integer $n \geq 1$ and let $S \subseteq \{0,1\}^n$ be a binary space whose points do not agree on a coordinate. Then there exists a subset $S' \subseteq S$, whose points do not agree on a coordinate, that is a duplication of the cocycle space of a projective geometry. As a consequence, if S has $GF(2)$ -rank r , then $\text{cuboid}(S)$ embeds one of $PG(0,2), \dots, PG(r-1,2)$.*

Proof Let A be a $0-1$ matrix whose rows a_1, \dots, a_m are linearly independent over $GF(2)$ such that $S = \{x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod{2}\}$. Clearly $\langle a_1, \dots, a_m \rangle$ is the orthogonal complement of S over $GF(2)$. As the points in S do not agree on a coordinate, $\langle a_1, \dots, a_m \rangle$ does not contain the unit vectors e_1, \dots, e_n (see Remark 29). Extend $\{a_1, \dots, a_m\}$ to a set $\{a_1, \dots, a_m, \dots, a_k\}$ of $0-1$ vectors such that

- (i) a_1, \dots, a_k are linearly independent over $GF(2)$,
- (ii) $\langle a_1, \dots, a_k \rangle$ does not contain e_1, \dots, e_n ,
- (iii) $\{a_1, \dots, a_k\}$ is maximal subject to (i)-(ii).

Let M be the binary matroid over ground set $[n]$ whose cycle space is $\langle a_1, \dots, a_k \rangle$. It follows from (ii) that M has no loop.

Claim 1 *For every pair of distinct elements f, g , there is a cycle of M of cardinality at most three containing f, g .*

Proof (of Claim) Let $a := \chi_{\{f,g\}} \in \{0,1\}^n$. If $a \in \langle a_1, \dots, a_k \rangle$, then $\{f, g\}$ is a cycle of M . Otherwise, a_1, \dots, a_k, a are linearly independent over $GF(2)$, so by (iii), $\langle a_1, \dots, a_k, a \rangle$ must contain one of e_1, \dots, e_n . Neither f nor g is a loop of M , so $\langle a_1, \dots, a_k \rangle$ must contain a vector b with three 1s such that $b_f = b_g = 1$. That is, M has a triangle containing f, g , as required. \diamond

Let $M' := \text{si}(M)$, the simplification of M . Then $\text{cocycle}(M)$ is a duplication of $\text{cocycle}(M')$. Claim 1 implies that every two elements of M' appear together in a triangle, so M' is a projective geometry by Proposition 38. Thus, $\text{cocycle}(M)$ is a duplication of the cocycle space of a projective geometry. Notice however that $\text{cocycle}(M) = \{x \in \{0,1\}^n : Bx \equiv \mathbf{0} \pmod{2}\}$ where B is the matrix whose rows are a_1, \dots, a_k . Subsequently, $S' := \text{cocycle}(M) \subseteq S$. As M has no loop, the points in S' do not agree on a coordinate, so S' is the desired set. \square

As an application of Proposition 39, Conjecture 36 can be rephrased as, *There exists an integer $\ell \geq 3$ such that every ideal tangled clutter “embeds a binary matroid” of rank at most ℓ .* As a second application of Proposition 39, we prove Conjecture 36 for $\ell = 3$ for the class of binary clutters.

Theorem 40 *Every ideal binary tangled clutter embeds $PG(0, 2)$, $PG(1, 2)$, or $PG(2, 2)$.*

Proof Let \mathcal{C} be a binary tangled clutter. By Proposition 33, \mathcal{C} is a duplication of a cuboid, say $\text{cuboid}(S)$ for some S containing $\mathbf{0}$. It suffices to show that $\text{cuboid}(S)$ embeds one of the three projective geometries. Note that $\text{cuboid}(S)$ is an ideal binary cuboid with $\tau(\text{cuboid}(S)) \geq 2$. By Theorem 15 and Remark 24, S is a cube-ideal binary space whose points do not agree on a coordinate. By Theorem 32, S has a subset of at most 3 points that do not agree on a coordinate. Let S' be the binary space generated by these points. Note that $S' \subseteq S$, the points in S' do not agree on a coordinate, and S' has $GF(2)$ -rank at most 3. By Proposition 39, $\text{cuboid}(S')$, and therefore $\text{cuboid}(S)$, embeds one of $PG(0, 2)$, $PG(1, 2)$, $PG(2, 2)$, as desired. \square

7 Two applications

In this final section, we discuss two applications of Theorem 40. The first application is a result in Graph Theory, that every bridgeless graph has a so-called *7-cycle 4-cover*.

Theorem 41 ([9], Proposition 6) *Every bridgeless graph has 7 cycles such that every edge is used in exactly 4 of the cycles.*

Proof Let G be a bridgeless graph. Applying Theorem 40 to $\text{cuboid}(\text{cycle}(G))$, we see that G has 8 cycles where every edge is used in exactly 4 of the cycles (some of the cycles may be repeated). Since one of the 8 cycles may be assumed to be \emptyset , G has 7 cycles such that each edge is in exactly 4 of the cycles. \square

For the next application, we prove that Conjecture 6 holds for ideal binary clutters with covering number two. We need the following ingredient.

Proposition 42 ([5]) *For every $k \in \mathbb{Z}_{\geq 0}$, $\text{cuboid}(\text{cocycle}(PG(k, 2)))$ has a $\frac{1}{2^k}$ -integral packing of value two.*

We are now ready for the second application.

Theorem 43 *Every ideal binary clutter \mathcal{C} with $\tau(\mathcal{C}) \geq 2$ has a $\frac{1}{4}$ -integral packing of value two.*

Proof Let \mathcal{C}' be a deletion minor of \mathcal{C} that is minimal subject to $\tau(\mathcal{C}') \geq 2$. Then \mathcal{C}' is an ideal binary tangled clutter, so by Theorem 40, \mathcal{C}' embeds one of $PG(0, 2)$, $PG(1, 2)$, $PG(2, 2)$. By Proposition 42, \mathcal{C}' , and therefore \mathcal{C} , has a $\frac{1}{4}$ -integral packing of value 2, as required. \square

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